On coefficient valuations of Eisenstein polynomials

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Abstract

Let $p \geq 3$ be a prime, let $n > m \geq 1$. Let π_n be the norm of $\zeta_{p^n} - 1$ under C_{p-1}, so that $\mathbf{Z}_{(p)}[\pi_n]|\mathbf{Z}_{(p)}$ is a purely ramified extension of discrete valuation rings of degree p^{n-1}. The minimal polynomial of π_n over $\mathbf{Q}(\pi_m)$ is an Eisenstein polynomial; we give lower bounds for its coefficient valuations at π_m. The function field analogue, as introduced by Carlitz and Hayes, is studied as well.

Contents

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0 Introduction

0.1 The problem

Suppose given a purely ramified extension of discrete valuation rings $S|R$, with maximal ideals generated by $s \in S$ and $r \in R$, respectively. In particular, $S = R[s]$. Let $L|K$ be the corresponding extension of the fields of fractions, and write $l = [L : K]$. The minimal polynomial $\mu_{s,K}(X) \in R[X]$ of s over K is an Eisenstein polynomial, that is, the valuation at r of its non-leading coefficients is ≥ 1, and the valuation of its constant term equals 1.

Thus $S/rS \simeq (R/rR)[X]/(X^l)$ is completely known. To obtain information about

$$S/r^m S \;\simeq\; (R/r^m R)[X]/(\mu_{s,K}(X))$$

for $m \geq 1$, however, we need to know better estimates for the coefficient valuations of the minimal polynomial $\mu_{s,K}(X)$. In short: how eisensteinian is it really?

The objective of this note is to give lower valuation bounds for the coefficients of $\mu_{s,K}(X)$ for certain subextensions $S|R$ of cyclotomic extensions, both in the number field and in the function field case. Moreover, the method we use enables us to relate the minimal polynomials for $T|R$ and for $S|R$ for iterated extensions $R \subseteq S \subseteq T$ of discrete valuation rings therein.

As an application, we mention the Wedderburn embedding of the twisted group ring (with trivial 2-cocycle)

$$S \wr C_l \;\overset{\omega}{\hookrightarrow}\; \mathrm{End}_R S \;\simeq\; R^{l\times l}\;,$$

where we assume $L|K$ to be galois with cyclic Galois group C_l. We have

$$\omega(S \wr C_l) \;\subseteq\; \Lambda \;:=\; \{f \in \mathrm{End}_R S \,:\, f(s^i S) \subseteq s^i S \text{ for } i \geq 0\} \;\subseteq\; \mathrm{End}_R S\;.$$

The image of s is the companion matrix of $\mu_{s,K}(X)$. To give a description of $\omega(S \wr C_l)$, it is convenient to be able to replace the image of s by the companion matrix of the 'optimal' Eisenstein polynomial $X^l - r$ modulo $s^j\Lambda$ for a suitable $j > 1$.

In this article, however, we restrict our attention to the minimal polynomial itself.

0.2 Results

0.2.1 The number field case

Let $p \geq 3$ be a prime, and let ζ_{p^n} denote a primitive p^nth root of unity over $\mathbf{Q}$ in such a way that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for all $n \geq 1$. Put $F_n = \mathbf{Q}(\zeta_{p^n})$ and let $E_n = \mathrm{Fix}_{C_{p-1}} F_n$, so $[E_n : \mathbf{Q}] = p^n$. Letting $\pi_n = \mathrm{N}_{F_n|E_n}(\zeta_{p_n} - 1) = \prod_{j\in[1,p-1]}(\zeta_{p^n}^{j^{p^{n-1}}} - 1)$, we have $E_n = \mathbf{Q}(\pi_n)$. In particular, $E_{m+i} = E_m(\pi_{m+i})$ for $m, i \geq 1$. We fix m and write

$$\mu_{\pi_{m+i},\,E_m}(X) \;=\; \sum_{j\in[0,p^i]} a_{i,j}X^j \;=\; X^{p^i} + \Big(\sum_{j\in[1,p^i-1]} a_{i,j}X^j\Big) - \pi_m \;\in\; \mathbf{Z}[\pi_m][X]\;.$$

Theorem (5.2, 5.4, 5.7).

(i) *We have* $p^i \mid ja_{i,j}$ *for* $j \in [0, p^i]$.

(i′) *If* $j < p^i(p-2)/(p-1)$, *then* $p^i\pi_m \mid ja_{i,j}$.

(ii) *We have* $a_{i,j} \equiv_{p^{i+1}} a_{i+\beta,p^\beta j}$ *for* $j \in [0, p^i]$ *and* $\beta \geq 1$.

(ii′) *If* $j < p^i(p-2)/(p-1)$, *then* $a_{i,j} \equiv_{p^{i+1}\pi_m} a_{i+\beta,p^\beta j}$ *for* $\beta \geq 1$.

(iii) *We have a unit* $a_{i,p^i-(p^i-p^\beta)/(p-1)} \,/\, p^{i-\beta} \in \mathbf{Z}_{(p)}[\pi_m]^*$ *for* $\beta \in [0, i-1]$.

(iv) *We have* $\mu_{\pi_n,\mathbf{Q}}(X) \equiv_{p^2} X^{p^{n-1}} + pX^{(p-1)p^{n-2}} - p$ *for* $n \geq 2$.

Assertion (iv) requires a computation of a trace. Such trace computations can be reformulated in terms of sums of $(p-1)$th roots of unity in $\mathbf{Q}_p$ (5.5). Essentially, one has to count the number of subsets of $\boldsymbol{\mu}_{p-1} \subseteq \mathbf{Q}_p$ of a given cardinality whose sum is of a given valuation at p. Not being able to go much beyond this reformulation, this seems to be a problem in its own right – see e.g. (5.8).

To prove (i, i′, ii, ii′), we proceed by induction. Assertions (i, i′) also result from the different $\mathfrak{D}_{\mathbf{Z}_{(p)}[\pi_{m+i}]|\mathbf{Z}_{(p)}[\pi_m]} = \left(\mu'_{\pi_{m+i},E_m}(\pi_{m+i})\right) = \left(p^i\pi_{m+i}^{p^i-1-(p^i-1)/(p-1)}\right)$. Moreover, using (ii), this different argument yields (iii). In the function field case below, this argument for (i, i′) fails, however, and we have to resort to induction.

Suppose $m = 1$. Let us call an index $j \in [1, p^i - 1]$ *exact,* if either $j < p^i(p-2)/(p-1)$ and $p^i\pi_m$ exactly divides $ja_{i,j}$, or $j \geq p^i(p-2)/(p-1)$ and p^i exactly divides $ja_{i,j}$. If $i = 1$ and e.g. $p \in \{3, 19, 29, 41\}$, then all indices $j \in [1, p-1]$ are exact. If $i \geq 2$, we might ask whether the number of non-exact indices j asymptotically equals p^{i-1} as $p \to \infty$.

0.2.2 The function field case

Let $p \geq 3$ be a prime, $\rho \geq 1$ and $r = p^\rho$. We write $\mathcal{Z} = \mathbf{F}_r[Y]$ and $\mathcal{Q} = \mathbf{F}_r(Y)$. We want to study a function field analogue over $\mathcal{Q}$ of the number field extension $\mathbf{Q}(\zeta_{p^n})|\mathbf{Q}$. Since 1 is the only p^nth root of unity in an algebraic closure $\bar{\mathcal{Q}}$, we have to proceed differently, following Carlitz [**1**] and Hayes [**5**]. First of all, the power operation of p^n on $\bar{\mathbf{Q}}$ becomes replaced by a module operation of f^n on $\bar{\mathcal{Q}}$, where $f \in \mathcal{Z}$ is an irreducible polynomial. The group of p^nth roots of unity

$$\boldsymbol{\mu}_{p^n} \;=\; \{\xi \in \bar{\mathbf{Q}} : \xi^{p^n} = 1\}$$

becomes replaced by the annihilator submodule

$$\boldsymbol{\lambda}_{f^n} \;=\; \{\xi \in \bar{\mathcal{Q}} : \xi^{f^n} = 0\}\ .$$

Instead of choosing a primitive p^nth root of unity ζ_{p^n}, i.e. a $\mathbf{Z}$-linear generator of that abelian group, we choose a $\mathcal{Z}$-linear generator θ_n of this $\mathcal{Z}$-submodule. A bit more precisely speaking, the element $\theta_n \in \bar{\mathcal{Q}}$ plays the role of $\vartheta_n := \zeta_{p^n} - 1 \in \bar{\mathbf{Q}}$. Now $\mathcal{Q}(\theta_n)|\mathcal{Q}$ is the function field analogue of $\mathbf{Q}(\vartheta_n)|\mathbf{Q}$. See also [**3,** sec. 2].

To state the result, let $f(Y) \in \mathcal{Z}$ be a monic irreducible polynomial and write $q = r^{\deg f}$. Let $\xi^Y := Y\xi + \xi^r$ define the $\mathcal{Z}$-linear Carlitz module structure on an algebraic closure $\bar{\mathcal{Q}}$, and choose a $\mathcal{Z}$-linear generator θ_n of $\operatorname{ann}_{f^n} \bar{\mathcal{Q}}$ in such a way that $\theta_{n+1}^f = \theta_n$ for all $n \geq 1$. We write $\mathcal{F}_n = \mathcal{Q}(\theta_n)$ and have $\operatorname{Gal}(\mathcal{F}_n/\mathbf{Q}) \simeq (\mathcal{Z}/f^n)^*$. Letting $\mathcal{E}_n = \operatorname{Fix}_{C_{q-1}} \mathcal{F}_n$, we get $[\mathcal{E}_n : \mathcal{Q}] = q^n$. Denoting $\varpi_n = \mathrm{N}_{\mathcal{F}_n|\mathcal{E}_n}(\theta_n) = \prod_{e \in (\mathcal{Z}/f)^*} \theta_n^{e^{q^{n-1}}}$, we have $\mathcal{E}_n = \mathcal{Q}(\varpi_n)$. In particular, $\mathcal{E}_{m+i} = \mathcal{E}_m(\varpi_{m+i})$ for $m, i \geq 1$. We fix m and write

$$\mu_{\varpi_{m+i}, \mathcal{E}_m}(X) = \sum_{j \in [0, q^i]} a_{i,j} X^j = X^{q^i} + \Big(\sum_{j \in [1, q^i - 1]} a_{i,j} X^j \Big) - \varpi_m \in \mathcal{Z}[\varpi_m][X] \, .$$

Let $v_q(j) := \min\{\alpha \geq 0 : \ q^\alpha \mid j \ \}$.

Theorem (6.6, 6.7, 6.9).

(i) *We have* $f^{i - v_q(j)} \mid a_{i,j}$ *for* $j \in [0, q^i]$.

(i′) *If* $j < q^i(q-2)/(q-1)$, *then* $f^{i - v_q(j)} \varpi_m \mid a_{i,j}$.

(ii) *We have* $a_{i,j} \equiv_{f^{i+1}} a_{i+\beta, q^\beta j}$ *for* $j \in [0, q^i]$ *and* $\beta \geq 1$.

(ii′) *If* $j < q^i(q-2)/(q-1)$, *then* $a_{i,j} \equiv_{f^{i+1}\varpi_m} a_{i+\beta, q^\beta j}$ *for* $\beta \geq 1$.

(iii) *We have a unit* $a_{i, q^i - (q^i - q^\beta)/(q-1)} \, / \, f^{i-\beta} \in \mathcal{Z}_{(f)}[\varpi_m]^*$ *for* $\beta \in [0, i-1]$.

(iv) *If* $f = Y$, *then* $\mu_{\varpi_{m+i}, \mathcal{E}_m}(X) \equiv_{Y^2} X^{q^i} + YX^{(q-1)q^{i-1}} - \varpi_m$.

A comparison of the assertions (iv) in the number field case and in the function field case indicates possible generalizations; we do not know what happens for $\mu_{\pi_{m+i}, E_m}(X)$ for $m \geq 2$ in the number field case; moreover, we do not know what happens for $f \neq Y$ in the function field case.

0.3 Notations and conventions

(o) Within a chapter, the lemmata, propositions etc. are numbered consecutively.

(i) For $a, b \in \mathbf{Z}$, we denote by $[a, b] := \{c \in \mathbf{Z} \ : \ a \leq c \leq b\}$ the interval in $\mathbf{Z}$.

(ii) For $m \in \mathbf{Z} \smallsetminus \{0\}$ and a prime p, we denote by $m[p] := p^{v_p(m)}$ the p-part of m, where v_p denotes the valuation of an integer at p.

(iii) If R is a discrete valuation ring with maximal ideal generated by r, we write $v_r(x)$ for the valuation of $x \in R \smallsetminus \{0\}$ at r, i.e. $x/r^{v_r(x)}$ is a unit in R. In addition, $v_r(0) := +\infty$.

(iv) Given an element x algebraic over a field K, we denote by $\mu_{x,K}(X) \in K[X]$ the minimal polynomial of x over K.

(v) Given a commutative ring A and an element $a \in A$, we sometimes denote the quotient by $A/a := A/aA$ — mainly if A plays the role of a base ring. For $b, c \in A$, we write $b \equiv_a c$ if $b - c \in aA$.

(vi) For an assertion X, which might be true or not, we let $\{X\}$ equal 1 if X is true, and equal 0 if X is false.

Throughout, let $p \geq 3$ be a prime.

1 A polynomial lemma

We consider the polynomial ring $\mathbf{Z}[X,Y]$.

Lemma 1.1 *For $k \geq 1$, we have* $(X+pY)^k \equiv_{k[p]\cdot p^2Y^2} X^k + kX^{k-1}pY$.

Since $\binom{k}{j} = k/j \cdot \binom{k-1}{j-1}$, we obtain for $j \geq 2$ that

$$\begin{aligned} v_p(p^j \binom{k}{j}) &\geq j + v_p(k) - v_p(j) \\ &\geq v_p(k) + 2 \,, \end{aligned}$$

where the second inequality follows from $j \geq 2$ if $v_p(j) = 0$, and from $j \geq p^{v_p(j)} \geq 3^{v_p(j)} \geq v_p(j) + 2$ if $v_p(j) \geq 1$.

Corollary 1.2 *For $k \geq 1$, we have* $(X+pY)^k \equiv_{k[p]\cdot pY} X^k$.

Corollary 1.3 *For $x, y \in \mathbf{Z}$ and $l \geq 1$ such that $x \equiv_{p^l} y$, and for $k \geq 1$, we have* $x^k \equiv_{k[p]\cdot p^l} y^k$.

Corollary 1.4 *For all $\alpha, \beta \geq 0$, we have* $(X+Y)^{p^{\beta+\alpha}} \equiv_{p^{\alpha+1}} (X^{p^\beta} + Y^{p^\beta})^{p^\alpha}$.

This being true for $\alpha = 0$, the assertion follows since $f(X,Y) \equiv_p g(X,Y)$ implies that $f(X,Y)^{p^\alpha} \equiv_{p^{\alpha+1}} g(X,Y)^{p^\alpha}$ by (1.2), where $f(X,Y)$, $g(X,Y) \in \mathbf{Z}[X,Y]$.

2 Consecutive purely ramified extensions

2.1 Setup

Let $T|S$ and $S|R$ be finite and purely ramified extensions of discrete valuation rings, of residue characteristic $\operatorname{char} R/rR = p$. The maximal ideals of R, S and T are generated by $r \in R$, $s \in S$ and $t \in T$, and the fields of fractions are denoted by $K = \operatorname{frac} R$, $L = \operatorname{frac} S$ and $M = \operatorname{frac} T$, respectively. Denote $m = [M : L]$ and $l = [L : K]$. We may and will assume $s = (-1)^{m+1}\mathrm{N}_{M|L}(t)$ and $r = (-1)^{l+1}\mathrm{N}_{L|K}(s)$.

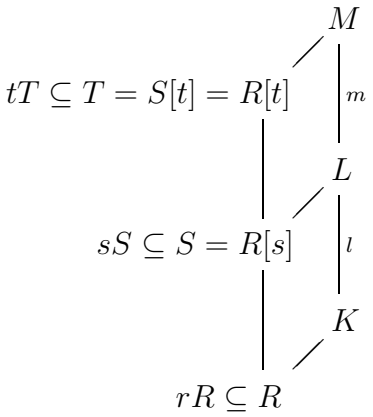
We have $S = R[s]$ with

$$\mu_{s,K}(X) \;=\; X^l + \Big(\sum_{j\in[1,l-1]} a_j X^j\Big) - r \;\in\; R[X] \,,$$

and $T = R[t]$ with

$$\mu_{t,K}(X) \;=\; X^{lm} + \Big(\sum_{j\in[1,lm-1]} b_j X^j\Big) - r \;\in\; R[X] \,.$$

Cf. [**9,** I.§7, prop. 18]. The situation can be summarized in the diagram



Note that $r \mid p$, and that for $z \in M$, we have $v_t(z) = mv_s(z) = mlv_r(z)$.

2.2 Characteristic 0

In this section, we assume $\operatorname{char} K = 0$. In particular, $\mathbf{Z}_{(p)} \subseteq R$.

Assumption 2.1 Suppose given $x, y \in T$ and $k \in [1, l-1]$ such that

(i) $p \mid y$ and $t^m \equiv_y s$,

(ii) $x \mid ja_j$ for all $j \in [1, l-1]$, and

(iii) $xr \mid ja_j$ for all $j \in [1, k-1]$.

Put $c := \gcd(xys^{k-1}, yls^{l-1}) \in T$.

Lemma 2.2 *Given* (2.1), *we have* $c \mid \mu_{s,K}(t^m)$.

We may decompose

$$\begin{aligned} \mu_{s,K}(t^m) &= \mu_{s,K}(t^m) - \mu_{s,K}(s) \\ &= (t^{ml} - s^l) + \left(\textstyle\sum_{j\in[1,k-1]} a_j(t^{mj} - s^j) \right) + \left(\textstyle\sum_{j\in[k,l-1]} a_j(t^{mj} - s^j) \right) . \end{aligned}$$

Now since $t^m = s + zy$ for some $z \in T$ by (2.1.i), we have

$$t^{mj} \overset{(1.1)}{\equiv}_{jy^2} s^j + js^{j-1}zy \equiv_{js^{j-1}y} s^j$$

for any $j \geq 1$, so that $s^{j-1} \mid r \mid p \mid y$ gives $t^{mj} \equiv_{js^{j-1}y} s^j$.

In particular, $yls^{l-1} \mid t^{ml} - s^l$.

Moreover, $xys^l \mid \sum_{j\in[1,k-1]} a_j(t^{mj} - s^j)$ by (2.1.iii).

Finally, $xys^{k-1} \mid \sum_{j\in[k,l-1]} a_j(t^{mj} - s^j)$ by (2.1.ii).

The following proposition will serve as inductive step in (3.2).

Proposition 2.3 *Given* (2.1), *we have* $t^{-j}c \mid b_j$ *if* $j \not\equiv_m 0$ *and* $t^{-j}c \mid (b_j - a_{j/m})$ *if* $j \equiv_m 0$ *for* $j \in [1, lm-1]$.

From (2.2) we take

$$\sum_{j\in[1,lm-1]} \left(b_j - \{j \equiv_m 0\}\, a_{j/m}\right) t^j \;=\; -\mu_{s,K}(t^m) \;\equiv_c\; 0\,.$$

Since the summands have pairwise different valuations at t, we obtain

$$\left(b_j - \{j \equiv_m 0\}\, a_{j/m}\right) t^j \;\equiv_c\; 0$$

for all $j \in [1, lm-1]$.

2.3 As an illustration: cyclotomic polynomials

For $n \geq 1$, we choose primitive roots of unity ζ_{p^n} over $\mathbf{Q}$ in such a manner that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$. We abbreviate $\vartheta_n = \zeta_{p^n} - 1$.

We shall show by induction on n that writing

$$\mu_{\vartheta_n,\mathbf{Q}}(X) \;=\; \Phi_{p^n}(X+1) \;=\; \sum_{j\in[0,p^{n-1}(p-1)]} d_{n,j} X^j$$

with $d_{n,j} \in \mathbf{Z}$, we have $p^{n-1} \mid j d_{n,j}$ for $j \in [0, p^{n-1}(p-1)]$, and even $p^n \mid j d_{n,j}$ for $j \in [0, p^{n-1}(p-2)]$.

This being true for $n = 1$ since $\Phi_p(X+1) = ((X+1)^p - 1)/X$, we assume it to be true for $n-1$ and shall show it for n, where $n \geq 2$. We apply the result of the previous section to $R = \mathbf{Z}_{(p)}$, $r = -p$, $S = \mathbf{Z}_{(p)}[\vartheta_{n-1}]$, $s = \vartheta_{n-1}$ and $T = \mathbf{Z}_{(p)}[\vartheta_n]$, $t = \vartheta_n$. In particular, we have $l = p^{n-2}(p-1)$ and $\mu_{s,K}(X) = \Phi_{p^{n-1}}(X+1)$; we have $m = p$ and $\mu_{t,L}(X) = (X+1)^p - 1 - \vartheta_{n-1}$; finally, we have $\mu_{t,K}(X) = \Phi_{p^n}(X+1)$.

We may choose $y = p\vartheta_n$, $x = p^{n-2}$ and $k = p^{n-2}(p-2)+1$ in (2.1). Hence $c = p^{n-1}\vartheta_n^{p^n - 2p^{n-1}+1}$. By (2.3), we obtain that $p^{n-1}\vartheta_n^{p^n-2p^{n-1}+1-j}$ divides $d_{n,j} - d_{n-1,j/p}$ if $j \equiv_p 0$ and that it divides $d_{n,j}$ if $j \not\equiv_p 0$. Since the coefficients in question are in R, we may draw the following conclusion.

(I) $\begin{cases} \text{If } j \equiv_p 0, \text{ then } p^n \mid d_{n,j} - d_{n-1,j/p} \text{ if } j \leq p^{n-1}(p-2), \\ \qquad \text{and } p^{n-1} \mid d_{n,j} - d_{n-1,j/p} \text{ if } j > p^{n-1}(p-2); \\ \text{if } j \not\equiv_p 0, \text{ then } p^n \mid d_{n,j} \text{ if } j \leq p^{n-1}(p-2), \\ \qquad \text{and } p^{n-1} \mid d_{n,j} \text{ if } j > p^{n-1}(p-2). \end{cases}$

By induction, this establishes the claim.

Using (1.4), assertion (I) also follows from the more precise relation

(II) $\qquad \Phi_{p^n}(X+1) - \Phi_{p^{n-1}}(X^p+1) \;\equiv_{p^n}\; X^{p^{n-1}(p-2)} \left((X^p+1)^{p^{n-2}} - (X+1)^{p^{n-1}}\right)$

for $n \geq 2$, which we shall show now. In fact,

$$\begin{aligned} & \left((X+1)^{p^n} - 1\right)\left((X^p+1)^{p^{n-2}} - 1\right) - \left((X^p+1)^{p^{n-1}} - 1\right)\left((X+1)^{p^{n-1}} - 1\right) \\ \overset{(1.4)}{\equiv}_{p^n} \;& \left((X^p+1)^{p^{n-1}} - 1\right)\left((X^p+1)^{p^{n-2}} - (X+1)^{p^{n-1}}\right) \\ \overset{(1.4)}{\equiv}_{p^n} \;& X^{p^n}\left((X^p+1)^{p^{n-2}} - (X+1)^{p^{n-1}}\right) \\ \overset{(1.4)}{\equiv}_{p^n} \;& X^{p^{n-1}(p-2)}\left((X^p+1)^{p^{n-2}} - (X+1)^{p^{n-1}}\right)\left((X+1)^{p^{n-1}} - 1\right)\left((X^p+1)^{p^{n-2}} - 1\right) \end{aligned}$$

and the result follows by division by the monic polynomial

$$\left((X+1)^{p^{n-1}}-1\right)\left((X^p+1)^{p^{n-2}}-1\right) .$$

Finally, we remark that writing $F_n(X) := \Phi_{p^n}(X+1) + X^{p^n-2p^{n-1}}(X+1)^{p^{n-1}}$, we can equivalently reformulate (II) to

(II′) $$F_n(X) \equiv_{p^n} F_{n-1}(X^p) .$$

2.4 Characteristic p

In this section, we assume $\operatorname{char} K = p$.

Assumption 2.4 Suppose given $x, y \in T$ and $k \in [1, l-1]$ such that

(i) $t^m \equiv_{ys} s$,

(ii) $x \mid a_j y^{j[p]}$ for all $j \in [1, l-1]$, and

(iii) $xr \mid a_j y^{j[p]}$ for all $j \in [1, k-1]$.

Let $c := \gcd(xs^k, y^{l[p]}s^l) \in T$.

Lemma 2.5 *Given* (2.4), *we have* $c \mid \mu_{s,K}(t^m)$.

We may decompose

$$\begin{aligned}\mu_{s,K}(t^m) &= \mu_{s,K}(t^m) - \mu_{s,K}(s) \\ &= (t^{ml} - s^l) + \left(\textstyle\sum_{j\in[1,k-1]} a_j(t^{mj} - s^j)\right) + \left(\textstyle\sum_{j\in[k,l-1]} a_j(t^{mj} - s^j)\right) .\end{aligned}$$

Now since $t^m \equiv_{ys} s$, we have $t^{mj} \equiv_{y^{j[p]}s^j} s^j$ for any $j \geq 1$.

In particular, $y^{l[p]}s^l \mid t^{ml} - s^l$.

Moreover, $xs^l \mid \sum_{j\in[1,k-1]} a_j(t^{mj} - s^j)$ by (2.4.iii).

Finally, $xs^k \mid \sum_{j\in[k,l-1]} a_j(t^{mj} - s^j)$ by (2.4.ii).

Proposition 2.6 *Given* (2.4), *we have* $t^{-j}c \mid b_j$ *if* $j \not\equiv_m 0$ *and* $t^{-j}c \mid (b_j - a_{j/m})$ *if* $j \equiv_m 0$ *for* $j \in [1, lm-1]$.

This follows using (2.5), cf. (2.3).

3 A tower of purely ramified extensions

Suppose given a chain

$$R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$$

of finite purely ramified extensions $R_{i+1}|R_i$, with maximal ideal generated by $r_i \in R_i$, of residue characteristic $\operatorname{char} R_i/r_iR_i = p$, with field of fractions $K_i = \operatorname{frac} R_i$, and of degree $[K_{i+1} : K_i] = p^\kappa = q$ for $i \geq 0$, where $\kappa \geq 1$ is an integer stipulated to be independent of i. We may and will suppose that $\mathrm{N}_{K_{i+1}|K_i}(r_{i+1}) = r_i$ for $i \geq 0$. We write

$$\mu_{r_i,K_0}(X) \;=\; X^{q^i} + \Big(\sum_{j\in[1,q^i-1]} a_{i,j}X^j \Big) - r_0 \;\in\; R_0[X] \;.$$

For $j \geq 1$, we denote $v_q(j) := \max\{\alpha \in \mathbf{Z}_{\geq 0} \;:\; j \equiv_{q^\alpha} 0\}$. That is, $v_q(j)$ is the largest integer below $v_p(j)/\kappa$. We abbreviate $g := (q-2)/(q-1)$.

Assumption 3.1 Suppose given $f \in R_0$ such that $r_i^{q-1}f \mid r_i^q - r_{i-1}$ for all $i \geq 0$. If $\operatorname{char} K_0 = 0$, then suppose $p \mid f \mid q$. If $\operatorname{char} K_0 = p$, then suppose $r_0 \mid f$.

Proposition 3.2 *Assume* (3.1).

(i) *We have* $f^{i-v_q(j)} \mid a_{i,j}$ *for* $i \geq 1$ *and* $j \in [1, q^i - 1]$.

(i′) *If* $j < q^ig$, *then* $f^{i-v_q(j)}r_0 \mid a_{i,j}$.

(ii) *We have* $a_{i,j} \equiv_{f^{i+1}} a_{i+\beta,q^\beta j}$ *for* $i \geq 1$, $j \in [1, q^i - 1]$ *and* $\beta \geq 1$.

(ii′) *If* $j < q^ig$, *then* $a_{i,j} \equiv_{f^{i+1}r_0} a_{i+\beta,q^\beta j}$ *for* $\beta \geq 1$.

Consider the case $\operatorname{char} K_0 = 0$. To prove (i, i′), we perform an induction on i, the assertion being true for $i = 1$ by (3.1). So suppose given $i \geq 2$ and the assertion to be true for $i - 1$. To apply (2.3), we let $R = R_0$, $r = r_0$, $S = R_{i-1}$, $s = r_{i-1}$, $T = R_i$ and $t = r_i$. Furthermore, we let $y = r_i^{q-1}f$, $x = f^{i-1}$ and $k = q^{i-1} - (q^{i-1} - 1)/(q - 1)$, so that (2.1) is satisfied by (3.1) and by the inductive assumption. We have $c = f^i r_i^{qk-1}$.

Consider $j \in [1, q^i - 1]$. If $j \not\equiv_q 0$, then (2.3) gives

$$v_{r_i}(a_{i,j}/f^i) \;\geq\; qk - 1 - j \;,$$

whence f^i divides $a_{i,j}$; f^i strictly divides $a_{i,j}$, if $j < q^ig$ since $0 < (qk-1) - q^ig = 1/(q-1) < 1$.

If $j \equiv_q 0$, then (2.3) gives

$$v_{r_i}((a_{i,j} - a_{i-1,j/q})/f^i) \;\geq\; qk - 1 - j \;,$$

whence f^i divides $a_{i,j} - a_{i-1,j/q}$; strictly, if $j < q^ig$. By induction, $f^{i-1-v_q(j/q)}$ divides $a_{i-1,j/q}$; strictly, if $j/q < q^{i-1}g$. But $a_{i-1,j/q} \equiv_{f^i} a_{i,j}$, and therefore $f^{i-v_q(j)}$ divides also $a_{i,j}$; strictly, if $j < q^ig$. This proves (i, i′).

The case $\beta = 1$ of (ii, ii′) has been established in the course of the proof of (i, i′). The general case follows by induction.

Consider the case $\operatorname{char} K_0 = p$. To prove (i, i′), we perform an induction on i, the assertion being true for $i = 1$ by (3.1). So suppose given $i \geq 2$ and the assertion to be true for

$i-1$. To apply (2.6), we let $R = R_0$, $r = r_0$, $S = R_{i-1}$, $s = r_{i-1}$, $T = R_i$ and $t = r_i$. Furthermore, we let $y = r_i^{-1} f$, $x = r_i^{-1} f^i$ and $k = q^{i-1} - (q^{i-1}-1)/(q-1)$, so that (2.4) is satisfied by (3.1) and by the inductive assumption. In fact, $xy^{-j[p]} = r_i^{j[p]-1} f^{i-j[p]}$ divides $f^{i-1-v_q(j)}$ both if $j \not\equiv_p 0$ and if $j \equiv_p 0$; in the latter case we make use of the inequality $p^{\alpha-1}(p-1) \geq \alpha+1$ for $\alpha \geq 1$, which needs $p \geq 3$. We obtain $c = f^i r_i^{qk-1}$.

Using (2.6) instead of (2.3), we may continue as in the former case to prove (i, i′), and, in the course of this proof, also (ii, ii′).

4 Galois descent of a divisibility

Let

$$\begin{array}{ccc} T & \xhookrightarrow{G} & \tilde{T} \\ \Big\uparrow{\scriptstyle m} & & \Big\uparrow{\scriptstyle m} \\ S & \xhookrightarrow{G} & \tilde{S} \end{array}$$

be a commutative diagram of finite, purely ramified extensions of discrete valuation rings. Let $s \in S$, $t \in T$, $\tilde{s} \in \tilde{S}$ and $\tilde{t} \in \tilde{T}$ generate the respective maximal ideals. Let $L = \operatorname{frac} S$, $M = \operatorname{frac} T$, $\tilde{L} = \operatorname{frac} \tilde{S}$ and $\tilde{M} = \operatorname{frac} \tilde{T}$ denote the respective field of fractions. We assume the extensions $M|L$ and $\tilde{L}|L$ to be linearly disjoint and $\tilde{M}$ to be the composite of M and $\tilde{L}$. Thus $m := [M:L] = [\tilde{M}:\tilde{L}]$ and $[\tilde{L}:L] = [\tilde{M}:M]$. We assume $\tilde{L}|L$ to be galois and identify $G := \operatorname{Gal}(\tilde{L}|L) = \operatorname{Gal}(\tilde{M}|M)$ via restriction. We may and will assume that $s = \mathrm{N}_{\tilde{L}|L}(\tilde{s})$, and that $t = \mathrm{N}_{\tilde{M}|M}(\tilde{t})$.

Lemma 4.1 *In $\tilde{T}$, the element $1 - \tilde{t}^m/\tilde{s}$ divides $1 - t^m/s$.*

Let $\tilde{d} = 1 - \tilde{t}^m/\tilde{s}$, so that $\tilde{t}^m = \tilde{s}(1-\tilde{d})$. We conclude

$$\begin{array}{rcl} t^m & = & \mathrm{N}_{\tilde{M}|M}(\tilde{t}^m) \\ & = & \mathrm{N}_{\tilde{L}|L}(\tilde{s}) \cdot \prod_{\sigma \in G}(1-\tilde{d}^\sigma) \\ & \equiv_{s\tilde{d}} & s\,. \end{array}$$

5 Cyclotomic number fields

5.1 Coefficient valuation bounds

For $n \geq 1$, we let ζ_{p^n} be a primitive p^nth root of unity over $\mathbf{Q}$. We make choices in such a manner that $\zeta_{p^n}^p = \zeta_{p^{n-1}}$ for $n \geq 2$. We denote $\vartheta_n = \zeta_{p^n} - 1$ and $F_n = \mathbf{Q}(\zeta_{p^n})$. Let $E_n = \operatorname{Fix}_{C_{p-1}} F_n$, so $[E_n : \mathbf{Q}] = p^{n-1}$. Let

$$\pi_n \;=\; \mathrm{N}_{F_n|E_n}(\vartheta_n) \;=\; \prod_{j \in [1,p-1]} (\zeta_{p^n}^{j p^{n-1}} - 1)\,.$$

The minimal polynomial $\mu_{\vartheta_n,F_{n-1}}(X) = (X+1)^p - \vartheta_{n-1} - 1$ shows that $\mathrm{N}_{F_n|F_{n-1}}(\vartheta_n) = \vartheta_{n-1}$, hence also $\mathrm{N}_{E_n|E_{n-1}}(\pi_n) = \pi_{n-1}$. Note that $\pi_1 = p$ and $E_1 = \mathbf{Q}$.

Let $\mathcal{O}$ be the integral closure of $\mathbf{Z}_{(p)}$ in E_n. Since $\mathrm{N}_{E_n|\mathbf{Q}}(\pi_n) = \pi_1 = p$, we have $\mathbf{Z}_{(p)}/p\mathbf{Z}_{(p)} \xrightarrow{\sim} \mathcal{O}/\pi_n\mathcal{O}$. In particular, the ideal $\pi_n\mathcal{O}$ in $\mathcal{O}$ is prime. Now $\pi_n^{p^{n-1}}\mathcal{O} = p\mathcal{O}$, since $\pi_n^{p^{n-1}}/p = \pi_n^{p^{n-1}}/\mathrm{N}_{E_n|\mathbf{Q}}(\pi_n) \in \mathbf{Z}_{(p)}[\vartheta_n]^* \cap E_n = \mathcal{O}^*$. Thus $\mathcal{O}$ is a discrete valuation ring, purely ramified of degree p^{n-1} over $\mathbf{Z}_{(p)}$, and so $\mathcal{O} = \mathbf{Z}_{(p)}[\pi_n]$ [**9,** I.§7, prop. 18]. In particular, $E_n = \mathbf{Q}(\pi_n)$.

Lemma 5.1 *For all $n \geq 2$, we have $\pi_n^p \equiv_{\pi_n^{p-1}p} \pi_{n-1}$.*

First of all, $\vartheta_n^p \equiv_{\vartheta_n p} \vartheta_{n-1}$ since $(X-1)^p - (X^p - 1)$ is divisible by $p(X-1)$ in $\mathbf{Z}[X]$. Letting $\tilde{T} = \mathbf{Z}_{(p)}[\vartheta_n]$ and $(\tilde{t}, \tilde{s}, t, s) = (\vartheta_n, \vartheta_{n-1}, \pi_n, \pi_{n-1})$, (4.1) shows that $1 - \vartheta_n^p/\vartheta_{n-1}$ divides $1 - \pi_n^p/\pi_{n-1}$. Therefore, $\vartheta_n p\, \vartheta_{n-1}^{-1}\pi_{n-1} \mid \pi_{n-1} - \pi_n^p$.

Now suppose given $m \geq 1$. To apply (3.2), we let $f = q = p$, $R_i = \mathbf{Z}_{(p)}[\pi_{m+i}]$ and $r_i = \pi_{m+i}$ for $i \geq 0$. We keep the notation

$$\mu_{\pi_{m+i},\,E_m}(X) \;=\; \mu_{r_i,\,K_0}(X) \;=\; X^{p^i} + \Big(\sum_{j\in[1,p^i-1]} a_{i,j}X^j\Big) - \pi_m \;\in\; R_0[X] \;=\; \mathbf{Z}_{(p)}[\pi_m][X]\,.$$

Theorem 5.2

(i) *We have $p^i \mid ja_{i,j}$ for $i \geq 1$ and $j \in [1, p^i - 1]$.*

(i′) *If $j < p^i(p-2)/(p-1)$, then $p^i\pi_m \mid ja_{i,j}$.*

(ii) *We have $a_{i,j} \equiv_{p^{i+1}} a_{i+\beta,p^\beta j}$ for $i \geq 1$, $j \in [1, p^i - 1]$ and $\beta \geq 1$.*

(ii′) *If $j < p^i(p-2)/(p-1)$, then $a_{i,j} \equiv_{p^{i+1}\pi_m} a_{i+\beta,p^\beta j}$.*

Assumption (3.1) is fulfilled by virtue of (5.1), whence the assertions follow by (3.2).

Example 5.3 For $p = 5$, $m = 1$ and $i = 2$, we have

$$\begin{aligned}\mu_{\pi_3,\mathbf{Q}}(X) \;=\;& X^{25} - 4\cdot 5^2X^{24} + 182\cdot 5^2X^{23} - 8\cdot 5^6X^{22} + 92823\cdot 5^2X^{21}\\ &- 6175454\cdot 5X^{20} + 12194014\cdot 5^2X^{19} - 18252879\cdot 5^3X^{18}\\ &+ 4197451\cdot 5^5X^{17} - 466901494\cdot 5^3X^{16} + 8064511079\cdot 5^2X^{15}\\ &- 4323587013\cdot 5^3X^{14} + 1791452496\cdot 5^4X^{13} - 113846228\cdot 5^6X^{12}\\ &+ 685227294\cdot 5^5X^{11} - 15357724251\cdot 5^3X^{10} + 2002848591\cdot 5^4X^9\\ &- 4603857997\cdot 5^3X^8 + 287207871\cdot 5^4X^7 - 291561379\cdot 5^3X^6\\ &+ 185467152\cdot 5^2X^5 - 2832523\cdot 5^3X^4 + 121494\cdot 5^3X^3 - 514\cdot 5^4X^2\\ &+ 4\cdot 5^4X - 5\,.\end{aligned}$$

Now $v_5(a_{3,22}) = 6 \neq 5 = v_5(a_{4,5\cdot 22})$, so the valuations of the coefficients considered in (5.2.ii) differ in general. This, however, does not contradict the assertion $a_{3,22} \equiv_{5^4} a_{4,\,5\cdot 22}$ from loc. cit.

5.2 A different proof of (5.2. i, i′) and some exact valuations

Let $m \geq 1$ and $i \geq 0$. We denote $R_i = \mathbf{Z}_{(p)}[\pi_{m+i}]$, $r_i = \pi_{m+i}$, $K_i = \operatorname{frac} R_i$, $\tilde{R}_i = \mathbf{Z}_{(p)}[\vartheta_{m+i}]$ and $\tilde{r}_i = \vartheta_{m+i}$. Denoting by $\mathfrak{D}$ the respective different [**9,** III.§3], we have $\mathfrak{D}_{\tilde{R}_i|\tilde{R}_0} = (p^i)$ and $\mathfrak{D}_{\tilde{R}_i|R_i} = (\tilde{r}_i^{p-2})$ [**9,** III.§3, prop. 13], whence

$$(*) \qquad \mathfrak{D}_{R_i|R_0} \;=\; \big(\mu'_{r_i,K_0}(r_i)\big) \;=\; \mathfrak{D}_{\tilde{R}_i|\tilde{R}_0}\mathfrak{D}_{\tilde{R}_0|R_0}\mathfrak{D}^{-1}_{\tilde{R}_i|R_i} \;=\; \Big(p^i r_i^{p^i-1-(p^i-1)/(p-1)}\Big)\ ,$$

cf. [**9,** III.§3, cor. 2]. Therefore, $p^i r_i^{p^i-1-(p^i-1)/(p-1)}$ divides $j a_{i,j} r_i^{j-1}$ for $j \in [1, p^i-1]$, and (5.2. i, i′) follow.

Moreover, since only for $j = p^i - (p^i-1)/(p-1)$ the valuations at r_i of $p^i r_i^{p^i-1-(p^i-1)/(p-1)}$ and $j a_{i,j} r_i^{j-1}$ are congruent modulo p^i, we conclude by $(*)$ that they are equal, i.e. that

$$a_{i,p^i-(p^i-1)/(p-1)} \,/\, p^i \;\in\; R_0^*\ .$$

Corollary 5.4 *We have*

$$a_{i,p^i-(p^i-p^\beta)/(p-1)} \,/\, p^{i-\beta} \;\in\; R_0^* \;=\; \mathbf{Z}_{(p)}[\pi_m]^* \qquad \textit{for} \quad \beta \in [0, i-1]\ .$$

This follows by (5.2.ii) from what we have just said.

E.g. in (5.3), 5^1 exactly divides $a_{2,25-5} = a_{2,20}$, and 5^2 exactly divides $a_{2,25-5-1} = a_{2,19}$.

5.3 Some traces

Let $\boldsymbol{\mu}_{p-1}$ denote the group of $(p-1)$st roots of unity in $\mathbf{Q}_p$. We choose a primitive $(p-1)$st root of unity $\zeta_{p-1} \in \boldsymbol{\mu}_{p-1}$ and may thus view $\mathbf{Q}(\zeta_{p-1}) \subseteq \mathbf{Q}_p$ as a subfield. Note that $[\mathbf{Q}(\zeta_{p-1}) : \mathbf{Q}] = \varphi(p-1)$, where φ denotes Euler's function. The restriction of the valuation v_p at p on $\mathbf{Q}_p$ to $\mathbf{Q}(\zeta_p)$, is a prolongation of the valuation v_p on $\mathbf{Q}$ to $\mathbf{Q}(\zeta_{p-1})$ (there are $\varphi(p-1)$ such prolongations).

Proposition 5.5 *For $n \geq 1$, we have*

$$\operatorname{Tr}_{E_n|\mathbf{Q}}(\pi_n) \;=\; p^n s_n - p^{n-1} s_{n-1}\ ,$$

where

$$s_n \;:=\; \frac{1}{p-1} \sum_{H \subseteq \boldsymbol{\mu}_{p-1}} (-1)^{\#H} \Big\{ v_p\big(\textstyle\sum_{\xi\in H} \xi\big) \geq n \Big\} \quad \textit{for} \quad n \geq 0\ .$$

We have $s_0 = 0$, and $s_n \in \mathbf{Z}$ for $n \geq 0$. The sequence $(s_n)_n$ becomes stationary at some minimally chosen $N_0(p)$. We have

$$N_0(p) \;\leq\; N(p) \;:=\; \max_{H \subseteq \boldsymbol{\mu}_{p-1}} \Big\{ v_p\big(\textstyle\sum_{\xi\in H} \xi\big) \colon \textstyle\sum_{\xi\in H} \xi \neq 0 \Big\} + 1\ .$$

An upper estimate for $N(p)$, hence for $N_0(p)$, is given in (5.12).

Proof of (5.5). For $j \in [1, p-1]$ the p-adic limits

$$\xi(j) \;:=\; \lim_{n\to\infty} j^{p^n}$$

exist since $j^{p^{n-1}} \equiv_{p^n} j^{p^n}$. They are distinct since $\xi(j) \equiv_p j$, and, thus, form the group $\boldsymbol{\mu}_{p-1} = \{\xi(j) \mid j \in [1, p-1]\}$. Using the formula

$$\mathrm{Tr}_{F_n|\mathbf{Q}}(\zeta_{p^n}^m) \;=\; p^n\big\{v_p(m) \geq n\big\} - p^{n-1}\big\{v_p(m) \geq n-1\big\}$$

and the fact that $j^{p^{n-1}} \equiv_{p^n} \xi(j)$, we obtain

$$\begin{aligned}
\mathrm{Tr}_{F_n|\mathbf{Q}}(\pi_n) \;&=\; \mathrm{Tr}_{F_n|\mathbf{Q}}\Big(\textstyle\prod_{j\in[1,\,p-1]}\big(1-\zeta_{p^n}^{j^{p^{n-1}}}\big)\Big)\\
&=\; \sum_{J\,\subseteq\,[1,\,p-1]} (-1)^{\#J}\,\mathrm{Tr}_{F_n|\mathbf{Q}}\Big(\zeta_{p^n}^{\sum_{j\in J} j^{p^{n-1}}}\Big)\\
&=\; \sum_{J\,\subseteq\,[1,\,p-1]} (-1)^{\#J}\Big(p^n\Big\{v_p\big(\textstyle\sum_{j\in J}\xi(j)\big) \geq n\Big\}\\
&\qquad\qquad - p^{n-1}\Big\{v_p\big(\textstyle\sum_{j\in J}\xi(j)\big) \geq n-1\Big\}\Big)\\
&=\; (p-1)(p^n s_n - p^{n-1}s_{n-1})\;,
\end{aligned}$$

whence

$$\mathrm{Tr}_{E_n|\mathbf{Q}}(\pi_n) \;=\; p^n s_n - p^{n-1}s_{n-1}\;.$$

Now $s_0 = 0 \in \mathbf{Z}$ by the binomial formula. Therefore, by induction, we conclude from $p^n s_n - p^{n-1}s_{n-1} \in \mathbf{Z}$ that $p^n s_n \in \mathbf{Z}$. Since $(p-1)s_n \in \mathbf{Z}$, too, we obtain $s_n \in \mathbf{Z}$.

As soon as $n \geq N(p)$, the conditions $v_p(\sum_{\xi\in H}\xi) \geq n$ and $v_p(\sum_{\xi\in H}\xi) = +\infty$ on $H \subseteq \boldsymbol{\mu}_{p-1}$ become equivalent, and we obtain

$$s_n \;=\; \frac{1}{p-1}\sum_{H\subseteq\boldsymbol{\mu}_{p-1}} (-1)^{\#H}\big\{\textstyle\sum_{\xi\in H}\xi = 0\big\}\,,$$

which is independent of n. Thus $N_0(p) \leq N(p)$.

Lemma 5.6 *We have $s_1 = 1$. In particular,* $\mathrm{Tr}_{E_2|\mathbf{Q}}(\pi_2) \;\equiv_{p^2}\; -p$.

Since $\mathrm{Tr}_{E_1|\mathbf{Q}}(\pi_1) = \mathrm{Tr}_{\mathbf{Q}|\mathbf{Q}}(p) = p$, and since $s_0 = 0$, we have $s_1 = 1$ by (5.5). The congruence for $\mathrm{Tr}_{E_2|\mathbf{Q}}(\pi_2)$ follows again by (5.5).

Corollary 5.7 *We have*

$$\mu_{\pi_n,\mathbf{Q}}(X) \;\equiv_{p^2}\; X^{p^{n-1}} + pX^{(p-1)p^{n-2}} - p$$

for $n \geq 2$.

By dint of (5.6), this ensues from (5.2. i′, ii).

Example 5.8 The last n for which we list s_n equals $N(p)$, except if there is a question mark in the next column. The table was calculated using Pascal ($p \leq 53$) and Magma ($p \geq 59$). In the last column, we list the upper bound for $N(p)$ calculated below (5.12).

s_n	$n=0$	1	2	3	4	5	6	7	upper bound for $N(p)$
$p=3$	0	1							
5	0	1							1
7	0	1							1
11	0	1	3						3
13	0	1	3						3
17	0	1	8	16					5
19	0	1	10	12					4
23	0	1	33	89	93				7
29	0	1	377	571	567				8
31	0	1	315	271	259				6
37	0	1	107	940	1296				9
41	0	1	6621	51693	18286	20186	20250		12
43	0	1	1707	4767	6921	6665			9
47	0	1	2250	272242	173355	181481	182361		16
53	0	1	71201	363798	1520045	1350049	1292229	1289925	18
59	0	1	1276	?					21
61	0	1	2516	?					12
67	0	1	407186	?					15
71	0	1	5816605	?					18
73	0	1	8370710	?					18
79	0	1	169135	?					18
83	0	1	632598	?					30
89	0	1	26445104	?					30
97	0	1	282789	?					24
101	0	1	25062002	?					31
103	0	1	56744199	?					25
107	0	1	1181268305	?					40
109	0	1	91281629	?					28
113	0	1	117774911422	?					37
127	0	1	6905447	?					28
131	0	1	2988330952791	?					37
137	0	1	1409600547	?					50
139	0	1	3519937121	?					34
149	0	1	25026940499	?					56
151	0	1	164670499159	?					31
157	0	1	51594129045351	?					38
163	0	1	288966887341	?					42
167	0	1	1205890070471	?					64
173	0	1	17802886165762	?					66
179	0	1	1311887715966	?					69
181	0	1	128390222739	?					38
191	0	1	233425263577158	?					57
193	0	1	306518196952028	?					51
197	0	1	347929949728221	?					66
199	0	1	9314622093145	?					48
211	0	1	12532938009082	?					39

So for example if $p = 31$, then $\mathrm{Tr}_{\mathbf{Q}(\pi_3)|\mathbf{Q}}(\pi_3) = 271 \cdot 31^3 - 315 \cdot 31^2$, whereas $\mathrm{Tr}_{\mathbf{Q}(\pi_7)|\mathbf{Q}}(\pi_7) = 259 \cdot 31^7 - 259 \cdot 31^6$. Moreover, $N_0(31) = N(31) = 4 \leq 6$.

Remark 5.9 Vanishing (resp. vanishing modulo a prime) of sums of roots of unity has been studied extensively. See e.g. **[2]**, **[6]**, where also further references may be found.

Remark 5.10 Neither do we know whether $s_n \geq 0$ nor whether $\mathrm{Tr}_{E_n|\mathbf{Q}}(\pi_n) \geq 0$ always hold. Moreover, we do not know a prime p for which $N_0(p) < N(p)$.

Remark 5.11 We calculated some further traces appearing in (5.2), using Maple and Magma.

For $p = 3$, $n \in [2, 10]$, we have $\mathrm{Tr}_{E_n|E_{n-1}}(\pi_n) = 3 \cdot 2$.

For $p = 5$, $n \in [2, 6]$, we have $\mathrm{Tr}_{E_n|E_{n-1}}(\pi_n) = 5 \cdot 4$.

For $p = 7$, $n \in [2, 5]$, we have $\mathrm{Tr}_{E_n|E_{n-1}}(\pi_n) = 7 \cdot 6$.

For $p = 11$, we have $\mathrm{Tr}_{E_2|E_1}(\pi_2) = 11 \cdot 32$, whereas

$$\begin{aligned}
&\mathrm{Tr}_{E_3|E_2}(\pi_3)\\
&= 22 \cdot (15 + \zeta^2 + 2\zeta^3 - \zeta^5 + \zeta^6 - 2\zeta^8 - \zeta^9 + 2\zeta^{14} - \zeta^{16} + \zeta^{18} - \zeta^{20} - 2\zeta^{24}\\
&\quad +2\zeta^{25} - 2\zeta^{26} - \zeta^{27} - \zeta^{31} + 2\zeta^{36} - \zeta^{38} + \zeta^{41} - \zeta^{42} - 2\zeta^{43} + 2\zeta^{47} - 3\zeta^{49}\\
&\quad -\zeta^{53} + \zeta^{54} + 2\zeta^{58} - \zeta^{60} - \zeta^{64} + \zeta^{67} + 2\zeta^{69} - \zeta^{71} - 2\zeta^{72} - \zeta^{75} - 2\zeta^{78}\\
&\quad +3\zeta^{80} - \zeta^{82} - \zeta^{86} + 2\zeta^{91} - \zeta^{93} - 2\zeta^{95} - 3\zeta^{97} + 2\zeta^{102} + \zeta^{103} - \zeta^{104} - \zeta^{108})\\
&= 22 \cdot 20144553545509393104 27^{-1} \cdot (34333871352527722810654\\
&\quad +13602724052675413182425 02\pi - 31857841148164445311437042\pi^2\\
&\quad +13573370840985597605965 8636\pi^3 - 83763613130017142371566453\pi^4\\
&\quad +20444806599344408104299252\pi^5 - 22963646312114426321689 32\pi^6\\
&\quad +11774374108386621881229 3\pi^7 - 27972584654252060850 93\pi^8\\
&\quad +27868038642441136108\pi^9 - 79170513243924842\pi^{10}) \,,
\end{aligned}$$

where $\zeta := \zeta_{11^2}$ and $\pi := \pi_2$.

5.4 An upper bound for $N(p)$

We view $\mathbf{Q}(\zeta_{p-1})$ as a subfield of $\mathbf{Q}_p$, and now, in addition, as a subfield of $\mathbf{C}$. Since complex conjugation commutes with the operation of $\mathrm{Gal}(\mathbf{Q}(\zeta_{p-1})|\mathbf{Q})$, we have $|\mathrm{N}_{\mathbf{Q}(\zeta_{p-1})|\mathbf{Q}}(x)| = |x|^{\varphi(p-1)}$ for $x \in \mathbf{Q}(\zeta_{p-1})$.

We abbreviate $\Sigma(H) := \sum_{\xi \in H} \xi$ for $H \subseteq \boldsymbol{\mu}_{p-1}$. Since $|\Sigma(H)| \leq p - 1$, we have $|\mathrm{N}_{\mathbf{Q}(\zeta_{p-1})|\mathbf{Q}}(\Sigma(H))| \leq (p-1)^{\varphi(p-1)}$. Hence, if $\Sigma(H) \neq 0$, then

$$v_p(\Sigma(H)) \;\leq\; v_p(\mathrm{N}_{\mathbf{Q}(\zeta_{p-1})|\mathbf{Q}}(\Sigma(H))) \;<\; \varphi(p-1) \,,$$

and therefore $N(p) \leq \varphi(p-1)$. We shall ameliorate this bound by a logarithmic term.

Proposition 5.12 *We have*

$$N(p) \;\leq\; \varphi(p-1)\left(1 - \frac{\log \pi}{\log p}\right) + 1$$

for $p \geq 5$.

It suffices to show that $|\Sigma(H)| \leq p/\pi$ for $H \subseteq \boldsymbol{\mu}_{p-1}$. We will actually show that

$$\max_{H \subseteq \boldsymbol{\mu}_{p-1}} |\Sigma(H)| \;=\; \frac{1}{\sin \frac{\pi}{p-1}} \,,$$

from which this inequality follows using $\sin x \geq x - x^3/6$ and $p \geq 5$.

Choose $H \subseteq \boldsymbol{\mu}_{p-1}$ such that $|\Sigma(H)|$ is maximal. Since $p-1$ is even, the $(p-1)$st roots of unity fall into pairs $(\eta, -\eta)$. The summands of $\Sigma(H)$ contain exactly one element of each such pair, since $|\Sigma(H)+\eta|^2 + |\Sigma(H)-\eta|^2 = 2|\Sigma(H)|^2 + 2$ shows that at least one of the inequalities $|\Sigma(H)+\eta| \leq |\Sigma(H)|$ and $|\Sigma(H)-\eta| \leq |\Sigma(H)|$ fails.

By maximality, replacing a summand η by $-\eta$ in $\Sigma(H)$ does not increase the value of $|\Sigma(H)|$, whence

$$|\Sigma(H)|^2 \ \geq \ |\Sigma(H) - 2\eta|^2 = |\Sigma(H)|^2 - 4\,\mathrm{Re}(\eta \cdot \overline{\Sigma(H)}) + 4\ ,$$

and thus

$$\mathrm{Re}(\eta \cdot \overline{\Sigma(H)}) \ \geq 1 \ > \ 0\ .$$

Therefore, the $(p-1)/2$ summands of $\Sigma(H)$ lie in one half-plane, whence the value of $|\Sigma(H)|$.

6 Cyclotomic function fields, after Carlitz and Hayes

6.1 Notation and basic facts

We shall give a brief review while fixing notation.

Let $\rho \geq 1$ and $r := p^\rho$. Write $\mathcal{Z} := \mathbf{F}_r[Y]$ and $\mathcal{Q} := \mathbf{F}_r(Y)$, where Y is an independent variable. We fix an algebraic closure $\bar{\mathcal{Q}}$ of $\mathcal{Q}$. The *Carlitz module structure* on $\bar{\mathcal{Q}}$ is defined by the $\mathbf{F}_r$-algebra homomorphism given on the generator Y as

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & \mathrm{End}_{\mathcal{Q}}\bar{\mathcal{Q}} \\ Y & \longmapsto & \Big(\xi \longmapsto \xi^Y \ := \ Y\xi + \xi^r\Big)\ . \end{array}$$

We write the module product of $\xi \in \bar{\mathcal{Q}}$ with $e \in \mathcal{Z}$ as ξ^e. For each $e \in \mathcal{Z}$, there exists a unique polynomial $P_e(X) \in \mathcal{Z}[X]$ that satisfies $P_e(\xi) = \xi^e$ for all $\xi \in \bar{\mathcal{Q}}$. In fact, $P_1(X) = X$, $P_Y(X) = YX + X^r$, and $P_{Y^{i+1}} = YP_{Y^i}(X) + P_{Y^i}(X^r)$ for $i \geq 1$. For a general $e \in \mathcal{Z}$, the polynomial $P_e(Y)$ is given by the according linear combination of these.

Note that $P_e(0) = 0$, and that $P_e'(X) = e$, whence $P_e(X)$ is separable, i.e. it decomposes as a product of distinct linear factors in $\bar{\mathcal{Q}}[X]$. Let

$$\boldsymbol{\lambda}_e \ = \ \mathrm{ann}_e\bar{\mathcal{Q}} \ = \ \{\xi \in \bar{\mathcal{Q}} \ : \ \xi^e = 0\} \ \subseteq \ \bar{\mathcal{Q}}$$

be the annihilator submodule. Separability of $P_e(X)$ shows that $\#\boldsymbol{\lambda}_e = \deg P_e(X) = r^{\deg e}$. Given a $\mathcal{Q}$-linear automorphism σ of $\bar{\mathcal{Q}}$, we have $(\xi^e)^\sigma = P_e(\xi)^\sigma = P_e(\xi^\sigma) = (\xi^\sigma)^e$. In particular, $\boldsymbol{\lambda}_e$ is stable under σ. Therefore, $\mathcal{Q}(\boldsymbol{\lambda}_e)$ is a Galois extension of $\mathcal{Q}$.

Since $\#\mathrm{ann}_{\tilde{e}}\boldsymbol{\lambda}_e = \#\boldsymbol{\lambda}_{\tilde{e}} = r^{\deg \tilde{e}}$ for $\tilde{e} \mid e$, we have $\boldsymbol{\lambda}_e \simeq \mathcal{Z}/e$ as $\mathcal{Z}$-modules. It is not possible, however, to distinguish a particular isomorphism.

We shall restrict ourselves to prime powers now. We fix a monic irreducible polynomial $f = f(Y) \in \mathcal{Z}$ and write $q := r^{\deg f}$. For $n \geq 1$, we let θ_n be a $\mathcal{Z}$-linear generator of $\boldsymbol{\lambda}_{f^n}$. We make our choices in such a manner that $\theta_{n+1}^f = \theta_n$ for $n \geq 1$. Note that $\mathcal{Z}[\boldsymbol{\lambda}_{f^n}] = \mathcal{Z}[\theta_n]$ since the elements of $\boldsymbol{\lambda}_{f^n}$ are polynomial expressions in θ_n.

Suppose given two roots $\xi, \tilde{\xi} \in \bar{\mathcal{Q}}$ of

$$\Psi_{f^n}(X) \ := \ P_{f^n}(X)/P_{f^{n-1}}(X) \ \in \ \mathcal{Z}[X] \ ,$$

i.e. $\xi, \tilde{\xi} \in \boldsymbol{\lambda}_{f^n} \smallsetminus \boldsymbol{\lambda}_{f^{n-1}}$. Since ξ is a $\mathcal{Z}$-linear generator of $\boldsymbol{\lambda}_{f^n}$, there is an $e \in \mathcal{Z}$ such that $\tilde{\xi} = \xi^e$. Since $\xi^e/\xi = P_e(X)/X|_{X=\xi} \in \mathcal{Z}[\theta_n]$, $\tilde{\xi}$ is a multiple of ξ in $\mathcal{Z}[\theta_n]$. Reversing the argument, we see that $\tilde{\xi}$ is in fact a unit multiple of ξ in $\mathcal{Z}[\theta_n]$.

Lemma 6.1 *The polynomial $\Psi_{f^n}(X)$ is irreducible.*

We have $\Psi_{f^n}(0) = \left.\dfrac{P_{f^n}(X)/X}{P_{f^{n-1}}(X)/X}\right|_{X=0} = f$. We decompose $\Psi_{f^n}(X) = \prod_{i\in[1,k]} F_i(X)$ in its distinct monic irreducible factors $F_i(X) \in \mathcal{Z}[X]$. One of the constant terms, say $F_j(0)$, is thus a unit multiple of f in $\mathcal{Z}$, while the other constant terms are units. Thus, all roots of $F_j(X)$ in $\mathcal{Q}[\theta_n]$ are non-units in $\mathcal{Z}[\theta_n]$, and the remaining roots of $\Psi_{f^n}(X)$ are units. But all roots of $\Psi_{f^n}(X)$ are unit multiples of each other. We conclude that $\Psi_{f^n}(X) = F_j(X)$ is irreducible.

By (6.1), $\Psi_{f^n}(X)$ is the minimal polynomial of θ_n over $\mathcal{Q}$. In particular, $[\mathcal{Q}(\theta_n) : \mathcal{Q}] = q^{n-1}(q-1)$, and so

$$\mathcal{Z}[\theta_n]\theta_n^{(q-1)q^{n-1}} \ = \ \mathcal{Z}[\theta_n]\mathrm{N}_{\mathcal{Q}(\theta_n)|\mathcal{Q}}(\theta_n) \ = \ \mathcal{Z}[\theta_n]f \ .$$

In particular, $\mathcal{Z}_{(f)}[\theta_n]$ is a discrete valuation ring with maximal ideal generated by θ_n, purely ramified of index $q^{n-1}(q-1)$ over $\mathcal{Z}_{(f)}$, cf. [**9,** I.§7, prop. 18]. There is a group isomorphism

$$\begin{array}{rcl} (\mathcal{Z}/f^n)^* & \xrightarrow{\sim} & \mathrm{Gal}(\mathcal{Q}(\theta_n)|\mathcal{Q}) \\ e & \longmapsto & (\theta_n \longmapsto \theta_n^e) \ , \end{array}$$

well defined since θ_n^e is a root of $\Psi_{f^n}(X)$, too; injective since θ_n generates $\boldsymbol{\lambda}_{f^n}$ over $\mathcal{Z}$; and surjective by cardinality.

Note that the Galois operation on $\mathcal{Q}(\theta_n)$ corresponding to $e \in (\mathcal{Z}/f^n)^*$ coincides with the module operation of e on the element θ_n, but not everywhere. For instance, if $f \neq Y$, then the Galois operation corresponding to Y sends 1 to 1, whereas the module operation of Y sends 1 to $Y+1$.

The discriminant of $\mathcal{Z}[\theta_n]$ over $\mathcal{Z}$ is given by $\Delta_{\mathcal{Z}[\theta_n]|\mathcal{Z}} = \mathrm{N}_{\mathcal{Q}(\theta_n)|\mathcal{Q}}(\Psi'_{f^n}(\theta_n))$ $= \mathrm{N}_{\mathcal{Q}(\theta_n)|\mathcal{Q}}\left(P'_{f^n}(\theta_n)/P_{f^{n-1}}(\theta_n)\right) = \mathrm{N}_{\mathcal{Q}(\theta_n)|\mathcal{Q}}\left(f^n/\theta_1\right) = f^{q^{n-1}(nq-n-1)}$.

Lemma 6.2 *The ring $\mathcal{Z}[\theta_n]$ is the integral closure of $\mathcal{Z}$ in $\mathcal{Q}(\theta_n)$.*

Let $e \in \mathcal{Z}$ be a monic irreducible polynomial different from f. Write $\mathcal{O}_0 := \mathcal{Z}_{(e)}[\theta_n]$ and let $\mathcal{O}$ be the integral closure of $\mathcal{O}_0$ in $\mathcal{Q}(\theta_n)$. Let

$$\begin{aligned} \mathcal{O}_0^+ &:= \{\xi \in \mathcal{Q}(\theta_n) \ : \ \mathrm{Tr}_{\mathcal{Q}(\theta_n)|\mathcal{Q}}(\xi\mathcal{O}_0) \subseteq \mathcal{Z}_{(e)}\} \\ \mathcal{O}^+ &:= \{\xi \in \mathcal{Q}(\theta_n) \ : \ \mathrm{Tr}_{\mathcal{Q}(\theta_n)|\mathcal{Q}}(\xi\mathcal{O}) \subseteq \mathcal{Z}_{(e)}\} \ . \end{aligned}$$

Then $\mathcal{O}_0 \subseteq \mathcal{O} \subseteq \mathcal{O}^+ \subseteq \mathcal{O}_0^+$. But $\mathcal{O}_0 = \mathcal{O}_0^+$, since the $\mathcal{Z}_{(e)}$-linear determinant of this embedding is given by the discriminant $\Delta_{\mathcal{Z}[\theta_n]|\mathcal{Z}}$, which is a unit in $\mathcal{O}_0$.

We resume.

Proposition 6.3 ([1],[5], cf. [3, p. 115]) *The extension $\mathcal{Q}(\theta_n)|\mathcal{Q}$ is galois of degree $[\mathcal{Q}(\theta_n) : \mathcal{Q}] = (q-1)q^{n-1}$, with Galois group isomorphic to $(\mathcal{Z}/f^n)^*$. The integral closure of $\mathcal{Z}$ in $\mathcal{Q}(\theta_n)$ is given by $\mathcal{Z}[\theta_n]$. We have $\mathcal{Z}[\theta_n]\theta_n^{[\mathcal{Q}(\theta_n):\mathcal{Q}]} = \mathcal{Z}[\theta_n]f$. In particular, θ_n is a prime element of $\mathcal{Z}[\theta_n]$, and the extension $\mathcal{Z}_{(f)}[\theta_n]|\mathcal{Z}_{(f)}$ of discrete valuation rings is purely ramified.*

6.2 Coefficient valuation bounds

Denote $\mathcal{F}_n = \mathcal{Q}(\theta_n)$. Let $\mathcal{E}_n = \mathrm{Fix}_{C_{q-1}}\mathcal{F}_n$, so $[\mathcal{E}_n : \mathcal{Q}] = q^{n-1}$. Let

$$\varpi_n \ = \ \mathrm{N}_{\mathcal{F}_n|\mathcal{E}_n}(\theta_n) \ = \ \prod_{e \in (\mathcal{Z}/f)^*} \theta_n^{e^{q^{n-1}}} \ .$$

The minimal polynomial $\mu_{\theta_n,\mathcal{F}_{n-1}}(X) = P_f(X) - \theta_{n-1}$ together with $X \,|\, P_f(X)$ shows that $\mathrm{N}_{\mathcal{F}_n|\mathcal{F}_{n-1}}(\theta_n) = \theta_{n-1}$, whence $\mathrm{N}_{\mathcal{E}_n|\mathcal{E}_{n-1}}(\varpi_n) = \varpi_{n-1}$. Note that $\varpi_1 = \prod_{e\in(\mathcal{Z}/f)^*} \theta_1^e = \Psi_f(0) = f$.

The extension $\mathcal{Z}_{(f)}[\varpi_n]$ is a discrete valuation ring with maximal ideal generated by ϖ_n, purely ramified of index q^{n-1} over $\mathcal{Z}_{(f)}$. In particular, $\mathcal{E}_n = \mathcal{Q}(\varpi_n)$.

Example 6.4 Let $r = 3$ and $f(Y) = Y^2 + 1$, so $q = 9$. A Magma calculation shows that

$$\begin{aligned} \varpi_2 &= \theta_2^{60} - Y\theta_2^{58} + Y^2\theta_2^{56} + (-Y^9-Y^3-Y)\theta_2^{42} + (Y^{10}+Y^4+Y^2+1)\theta_2^{40} \\ &+ (-Y^{11}-Y^5-Y^3+Y)\theta_2^{38} + (-Y^6-Y^4-Y^2)\theta_2^{36} + (Y^7+Y^5+Y^3+Y)\theta_2^{34} \\ &+ (-Y^8-Y^6+Y^4-Y^2-1)\theta_2^{32} + (-Y^5+Y^3-Y)\theta_2^{30} + (Y^{18}-Y^{12}-Y^{10}+Y^6-Y^4+Y^2)\theta_2^{24} \\ &+ (-Y^{19}+Y^{13}+Y^{11}+Y^9-Y^7+Y^5+Y)\theta_2^{22} \\ &+ (Y^{20}-Y^{14}-Y^{12}+Y^{10}+Y^8-Y^6-Y^4+Y^2+1)\theta_2^{20} \\ &+ (-Y^{15}-Y^{13}-Y^{11}-Y^9+Y^7+Y^5-Y^3)\theta_2^{18} + (Y^{16}+Y^{14}+Y^{12}-Y^{10}-Y^8-Y^2)\theta_2^{16} \\ &+ (-Y^{17}-Y^{15}+Y^{13}+Y^{11}+Y^7+Y^5-Y^3+Y)\theta_2^{14} \\ &+ (-Y^{14}-Y^{12}+Y^{10}-Y^8-Y^6-Y^4+Y^2+1)\theta_2^{12} + (-Y^{13}+Y^{11}-Y^7+Y^3)\theta_2^{10} \\ &+ (Y^{14}-Y^{12}-Y^{10}+Y^6+Y^4)\theta_2^8 + (-Y^{11}-Y^7+Y^5+Y^3+Y)\theta_2^6 + (Y^8+Y^6+Y^2+1)\theta_2^4 \ . \end{aligned}$$

With regard to section 6.4, we remark that $\varpi_2 \neq \pm\,\theta_2^{q-1}$.

Lemma 6.5 *For all $n \geq 2$, we have* $\varpi_n^q \equiv_{\varpi_n^{q-1} f} \varpi_{n-1}$.

We claim that $\theta_n^q \equiv_{\theta_n f} \theta_{n-1}$. In fact, the non-leading coefficients of the Eisenstein polynomial $\Psi_f(X)$ are divisible by f, so that the congruence follows by $\theta_{n-1} - \theta_n^q = P_f(\theta_n) - \theta_n^q = \theta_n(\Psi_f(\theta_n) - \theta_n^{q-1})$. Letting $\tilde{T} = \mathcal{Z}_{(f)}[\theta_n]$ and $(\tilde{t}, \tilde{s}, t, s) = (\theta_n, \theta_{n-1}, \varpi_n, \varpi_{n-1})$, (4.1) shows that $1 - \theta_n^q/\theta_{n-1}$ divides $1 - \varpi_n^q/\varpi_{n-1}$. Therefore, $\theta_n f \theta_{n-1}^{-1} \varpi_{n-1} \mid \varpi_{n-1} - \varpi_n^q$.

Now suppose given $m \geq 1$. To apply (3.2), we let $R_i = \mathcal{Z}_{(f)}[\varpi_{m+i}]$ and $r_i = \varpi_{m+i}$ for $i \geq 0$. We continue to denote

$$(\#) \qquad \begin{aligned} \mu_{\varpi_{m+i}, \mathcal{E}_m}(X) \;=\; \mu_{r_i, K_0}(X) \;&=\; X^{q^i} + \Big(\textstyle\sum_{j \in [1, q^i - 1]} a_{i,j} X^j \Big) - \varpi_m \\ &\in\; R_0[X] \;=\; \mathcal{Z}_{(f)}[\varpi_m][X] \,, \end{aligned}$$

and $v_q(j) = \max\{\alpha \in \mathbf{Z}_{\geq 0} \;:\; j \equiv_{q^\alpha} 0 \,\}$.

Theorem 6.6

(i) *We have $f^{i - v_q(j)} \mid a_{i,j}$ for $i \geq 1$ and $j \in [1, q^i - 1]$.*

(i′) *If $j < q^i(q-2)/(q-1)$, then $f^{i - v_q(j)} \varpi_m \mid a_{i,j}$.*

(ii) *We have $a_{i,j} \equiv_{f^{i+1}} a_{i+\beta, q^\beta j}$ for $i \geq 1$, $j \in [1, q^i - 1]$ and $\beta \geq 1$.*

(ii′) *If $j < q^i(q-2)/(q-1)$, then $a_{i,j} \equiv_{f^{i+1} \varpi_m} a_{i+\beta, q^\beta j}$ for $\beta \geq 1$.*

Assumption (3.1) is fulfilled by virtue of (6.5), whence the assertions follow by (3.2).

6.3 Some exact valuations

Let $m \geq 1$ and $i \geq 0$. We denote $R_i = \mathcal{Z}_{(f)}[\varpi_{m+i}]$, $r_i = \varpi_{m+i}$, $K_i = \operatorname{frac} R_i$, $\tilde{R}_i = \mathcal{Z}_{(f)}[\theta_{m+i}]$ and $\tilde{r}_i = \theta_{m+i}$. We obtain $\mathfrak{D}_{\tilde{R}_i | \tilde{R}_0} = (f^i)$ and $\mathfrak{D}_{\tilde{R}_i | R_i} = (\tilde{r}_i^{q-2})$ [**9,** III.§3, prop. 13], whence

$$(**) \qquad \mathfrak{D}_{R_i | R_0} \;=\; \big(\mu'_{r_i, K_0}(r_i) \big) \;=\; \Big(f^i r_i^{q^i - 1 - (q^i - 1)/(q-1)} \Big) \,.$$

Therefore, $f^i r_i^{q^i - 1 - (q^i - 1)/(q-1)}$ divides $j a_{i,j} r_i^{j-1}$ for $j \in [1, q^i - 1]$, which is an empty assertion if $j \equiv_p 0$. Thus (6.6.i, i′) do not follow entirely.

However, since only for $j = q^i - (q^i - 1)/(q-1)$ the valuations at r_i of $f^i r_i^{q^i - 1 - (q^i - 1)/(q-1)}$ and $j a_{i,j} r_i^{j-1}$ are congruent modulo q^i, we conclude by $(**)$ that they are equal, i.e. that

$$a_{i, q^i - (q^i - 1)/(q-1)} \,/\, f^i \;\in\; R_0^* \,.$$

Corollary 6.7 *We have*

$$a_{i, q^i - (q^i - q^\beta)/(q-1)} \,/\, f^{i-\beta} \;\in\; R_0^* \;=\; \mathcal{Z}_{(f)}[\varpi_m]^* \qquad \textit{for} \quad \beta \in [0, i-1] \,.$$

This follows by (6.6.ii) from what we have just said.

6.4 A simple case

Suppose that $f(Y) = Y$ and $m \geq 1$. Note that

$$\varpi_{m+1} \;=\; \prod_{e \in \mathbf{F}_q^*} \theta_{m+1}^e \;=\; \prod_{e \in \mathbf{F}_q^*} e\theta_{m+1} \;=\; -\theta_{m+1}^{q-1} \; .$$

Lemma 6.8 *We have*

$$\mu_{\varpi_{m+1},\mathcal{E}_m}(X) \;=\; -\varpi_m + \sum_{j \in [1,q]} Y^{q-j} X^j \; .$$

Using the minimal polynomial $\mu_{\theta_{m+1},\mathcal{F}_m}(X) = P_Y(X) - \theta_m = X^q + YX - \theta_m$, we get

$$\begin{array}{ll} & -\varpi_m + \sum_{j \in [1,q]} Y^{q-j} \varpi_{m+1}^j \\ = & \theta_m^{q-1} + (Y^{q+1} - \theta_{m+1}^{q^2-1})/(Y + \theta_{m+1}^{q-1}) - Y^q \\ = & (Y\theta_m^{q-1}\theta_{m+1} + \theta_m^{q-1}\theta_{m+1}^q - \theta_{m+1}^{q^2} - Y^q\theta_{m+1}^q)/(\theta_{m+1}(Y + \theta_{m+1}^{q-1})) \\ = & 0 \; . \end{array}$$

Corollary 6.9 *Let $m, i \geq 1$. We have*

$$\mu_{\varpi_{m+i},\mathcal{E}_m}(X) \;\equiv_{Y^2}\; X^{q^i} + YX^{(q-1)q^{i-1}} - \varpi_m \; .$$

This follows from (6.8) using (6.6.ii).

Remark 6.10 (6.8) also holds if $p = 2$.

Conjecture 6.11 Let $m, i \geq 1$. We use the notation of (#) above, now in the case $f(Y) = Y$. For $j \in [1, q^i]$, we write $q^i - j = \sum_{k \in [0,i-1]} d_k q^k$ with $d_k \in [0, q-1]$. Consider the following conditions.

(i) There exists $k \in [0, i-2]$ such that $d_{k+1} < d_k$.

(ii) There exists $k \in [0, i-2]$ such that $v_p(d_{k+1}) > v_p(d_k)$.

If (i) or (ii) holds, then $a_{i,j} = 0$. If neither (i) nor (ii) holds, then

$$v_{\varpi_m}(a_{i,j}) \;=\; q^{m-1} \cdot \sum_{k \in [0,i-1]} d_k \; .$$

Remark 6.12 We shall compare (6.7) with (6.11). If $j = q^i - (q^i - q^\beta)/(q-1)$ for some $\beta \in [0, i-1]$, then $q^i - j = q^{i-1} + \cdots + q^\beta$. Hence $\sum_{k \in [0,i-1]} d_k = i - \beta$, and so according to (6.11), $v_{\varpi_m}(a_{i,j})$ should equal $q^{m-1}(i-\beta)$, which is in fact confirmed by (6.7).

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